ELSEVIER Journal of Computational and Applied Mathematics 81 (1997) 145–163

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Approximation of circular arcs by Bézier curves

Young Joon Ahn, Hong Oh Kim*, [1]

KAIST, Department of Mathematics, Yusong-Gu, 373-1 Kusong-Dong, Taejon, 305-701 South Korea

Received 12 June 1996; received in revised form 15 February 1997

Abstract

For the circular arc of angle $0<\alpha<\pi$ we present the explicit form of the best GC^3 quartic approximation and the best GC^2 quartic approximations of various types, and give the explicit form of the Hausdorff distance between the circular arc and the approximate Bézier curves for each case. We also show the existence of the GC^4 quintic approximations to the arc, and find the explicit form of the best GC^3 quintic approximation in certain constraints and their distances from the arc. All approximations we construct in this paper have the optimal order of approximation, twice of the degree of approximate Bézier curves.

Keywords: Circular arcs; Quartic Bézier; Quintic Bézier; Best approximation; Geometric continuity; Hausdorff distance

1. Introduction

Optimal approximation of parametric curves and surfaces is one of the most important problems in CAGD. As the complexity of such approximation is high, circle approximation has to be the key point to address the possibilities of better approximation techniques. For the planar curves, de Boor et al. [1] found the GC^2 cubic approximation having its optimal approximation order six. For the better approximation of circular arc by cubic Bézier curves, Dokken et al. [4] gave the curvature continuous cubic approximation and Goldapp [9] presented the best GC^k cubic approximations for $k=0,1,2$, whose approximation order is six. Mørken [14] also suggested the best approximation to the circular arc from the quadratic Bézier curves with approximation order four.

We extend the previous works to the quartic and quintic Bézier curve approximation of circular arc. The algebraic manipulations are expectedly more involved but are fortunately manageable. More explicitly, we give the best quartic GC^3 approximation to the arc of angle $0<\alpha<\pi$, find its explicit

* Corresponding author.

[1] Partially supported by TGRC-KOSEF.

0377-0427/97/$17.00 © 1997 Elsevier Science B.V. All rights reserved
PII S0377-0427(97)00037-X

Table 1
The Hausdorff distance between the circular arc $\mathbf{q}$ of angle $\alpha=\pi/2$ and the best approximate Bézier curve $\mathbf{b}$ (see Sections 3 and 4)

Degree	Approximation type	Best approximation	$\mathscr{H}(\mathbf{q},\mathbf{b})$
Cubic	GC^2		1.96×10^{-3}
Quartic	$GC^3=$ Best GC^{2+}	$\mathbf{b}_{u_3}$ ($u_3=0.402437$)	3.50×10^{-5}
	Best GC^{2-}	$\mathbf{b}_{\mu_2}$ ($\mu_2=0.402599$)	3.55×10^{-6}
Quintic	GC^4	$\mathbf{b}_{v_1}$ ($v_1=0.285819$)	3.50×10^{-5}
		$\mathbf{b}_{v_2}$ ($v_2=0.318858$)	3.68×10^{-7}
	Best GC^{3+}	$\mathbf{b}_{v_2}$ ($v_2=0.318892$)	2.95×10^{-8}

form which is easy to use and its distance from the arc, and we show that it has the optimal approximation order eight. We also find the quintic GC^4 approximation to the arc and its distance, and show that it has its optimal approximation order ten. (Refer to [1, 3, 8, 11, 12, 16].)

In CAD or CAGD, we have to deal with the signed error function $\psi(t)$ which is a signed distance in the normal direction from each point of the arc to the approximate Bézier curve. We find the explicit forms of the best quartic GC^{2+} or GC^{2-} approximation for the case where ψ is positive or negative, respectively, and the explicit form of the best quintic GC^{3+} approximation for the case where ψ is positive. (Refer to [2, 4–6].) We also find their Hausdorff distances and their approximation orders.

As an illustration, we present the Hausdorff distances between the circular arc $\mathbf{q}$ of angle $\alpha=\frac{1}{2}\pi$ and the best approximate Bézier curves $\mathbf{b}$ in Table 1. The cubic approximation was proposed by de Boor [1], and the best quartic GC^3 and $GC^{2\pm}$ approximations, and the best quintic GC^4 and GC^{3+} approximations are the results of our methods in this paper.

In Section 2, we give the definitions and basic facts which are needed in the geometric approximation theory. We present the quartic approximations to the circular arcs in Section 3, and the quintic approximations in Section 4. We summarize our works in Section 5.

2. Basic facts for the circular arcs

We assume that the circular arc of angle $0<\alpha<\pi$ to be approximated by a Bézier curve is a portion of the unit circle and that it starts at the point $[1,0]^{\mathrm{T}}$ and ends at the point $[\cos\alpha,\sin\alpha]^{\mathrm{T}}$. We parametrize this circular arc $\mathbf{q}:[0,1]\to\mathbb{R}^2$ by

$$\mathbf{q}(s):=\begin{bmatrix}\cos(\alpha s)\\ \sin(\alpha s)\end{bmatrix},\quad 0\leqslant s\leqslant 1. \tag{2.1}$$

Let $\mathbf{b}:[0,1]\to\mathbb{R}^2$ be the Bézier curve of degree n with its control points $\mathbf{b}_i:=[x_i,y_i]^{\mathrm{T}}$, $i=0,1,2,\ldots,n$. The Bézier curve can be parametrized as

$$\mathbf{b}(t):=\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}:=\begin{bmatrix}\sum_{i=0}^{n}x_iB_{n,i}(t)\\ \sum_{i=0}^{n}y_iB_{n,i}(t)\end{bmatrix},\quad 0\leqslant t\leqslant 1, \tag{2.2}$$

where the Bernstein polynomial $B_{n,i}(t)$ is given by

$$B_{n,i}(t)=\binom{n}{i}t^i(1-t)^{n-i}.$$

Since the source curve $\mathbf{q}(s)$ is symmetric with respect to the symmetric axis $y=x\tan(\frac{1}{2}\alpha)$, we find the approximations to $\mathbf{q}$ from the Bézier curves which are symmetric with respect to the same axis.

Definition 2.1. Let k be a nonnegative integer and $t_0\in\{0,1\}$. Two C^k curves $\mathbf{p}$ and $\mathbf{q}$ have *contact of order* k at t_0 if $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are regular near t_0 and there are C^k reparametrizations τ_1 and τ_2 such that $\tau_i(t_0)=t_0$ $(i=1,2)$, $\tau_1'(t_0)\tau_2'(t_0)>0$ and

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}\mathbf{p}(\tau_1(t))\bigg|_{t=t_0}=\frac{\mathrm{d}^i}{\mathrm{d}t^i}\mathbf{q}(\tau_2(t))\bigg|_{t=t_0}\quad \text{for } i=0,\ldots,k. \tag{2.3}$$

(Refer to [6, 10].)

If $\mathbf{p}$ has contact of order k at $t=0,1$ to the given curve $\mathbf{q}$, then we call $\mathbf{p}$ a GC^k *approximation* (or GC^k *interpolation*) to $\mathbf{q}$. In this paper, each approximate Bézier curve $\mathbf{b}$ is at least a GC^0 approximation to the circular arc $\mathbf{q}$. Thus $\mathbf{b}$ satisfies

$$\mathbf{b}_0=\mathbf{b}(0)=\mathbf{q}(0)=\begin{bmatrix}1\\0\end{bmatrix}\quad\text{and}\quad \mathbf{b}_n=\mathbf{b}(1)=\mathbf{q}(1)=\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}. \tag{2.4}$$

We introduce a function $\psi(t):=x(t)^2+y(t)^2-1$ (Refer to [4].) and a closed subset W of $\mathbb{R}^2$ such that

$$W=\{(r\cos\theta,\, r\sin\theta)\in\mathbb{R}^2: 0\leqslant\theta\leqslant\alpha \text{ and } r\geqslant\delta\},$$

for a sufficiently small $\delta>0$. We also define the uniform norm of ψ as

$$\|\psi(t)\|_{[0,1]}:=\max_{t\in[0,1]}|\psi(t)|,$$

and define $\mathcal{W}_n^k$ as the class of all Bézier curves $\mathbf{b}(t)$ of degree $n\geqslant 2$ with contact order $k\geqslant 0$ at each end points of $\mathbf{q}(s)$ such that $\mathbf{b}(t)\in W$ for all $t\in[0,1]$. Then $\mathcal{W}_n^0\supset\mathcal{W}_n^1\supset\mathcal{W}_n^2\supset\cdots$ and $\mathcal{W}_2^k\subset\mathcal{W}_3^k\subset\mathcal{W}_4^k\subset\cdots$. We put $\mathcal{W}_n^{k+}:=\{\mathbf{b}\in\mathcal{W}_n^k:\psi_{\mathbf{b}}(t)\geqslant 0 \text{ for all } t\in[0,1]\}$ and $\mathcal{W}_n^{k-}:=\{\mathbf{b}\in\mathcal{W}_n^k: \psi_{\mathbf{b}}(t)\leqslant 0 \text{ for all } t\in[0,1]\}$. (Refer to [2, 5, 6].)

Lemma 2.2. *The Bézier curve* $\mathbf{b}\in\mathcal{W}_n^0$ *satisfies*

$$\frac{\mathrm{d}^i\psi(t)}{\mathrm{d}t^i}\bigg|_{t=0}=0\quad \textit{for } i=1,\ldots,k \tag{2.5}$$

if and only if $\mathbf{b}$ *is the* GC^k *approximation to the circular arc* $\mathbf{q}$.

Proof. Since ψ is symmetric, $\mathbf{b}\in\mathcal{W}_n^0$ satisfies Eq. (2.5) if and only if its signed error function $\psi(t)$ has the zeros of multiplicity $k+1$ at $t=0$ and $t=1$. Since $\psi(t)=0$ is an implicit equation of the unit circle $\mathbf{q}$, $\psi(t)$ has the zeros of multiplicity $k+1$ at $t=0$ and $t=1$ if and only if the two curves $\mathbf{b}$ and $\mathbf{q}$ have contact of order k at the points. Thus we obtain the assertion. □

For $\mathbf{b}\in\mathscr{W}_n^{k+}$ or $\mathbf{b}\in\mathscr{W}_n^{k-}$, the Hausdorff distance $\mathscr{H}(\mathbf{q},\mathbf{b})$ has a nice form as in the following proposition.

Proposition 2.3. *For each n and k, if* $\mathbf{b}\in\mathscr{W}_n^{k+}$, *then the Hausdorff distance* $\mathscr{H}(\mathbf{q},\mathbf{b})$ *between two curves* $\mathbf{q}$ *and* $\mathbf{b}$ *is*

$$\mathscr{H}(\mathbf{q},\mathbf{b})=\sqrt{\|\psi\|_{[0,1]}+1}-1, \tag{2.6}$$

and if $\mathbf{b}\in\mathscr{W}_n^{k-}$, *then*

$$\mathscr{H}(\mathbf{q},\mathbf{b})=1-\sqrt{1-\|\psi\|_{[0,1]}}. \tag{2.7}$$

Proof. Since $\mathbf{b}(t)$ lies in W for all $t\in[0,1]$ and its image is compact, $\mathscr{H}(\mathbf{b},\mathbf{q})=\max_{t\in[0,1]}|\|\mathbf{b}(t)\|-1|$. For $\mathbf{b}\in\mathscr{W}_n^{k+}$, Eq. (2.6) follows from

$$\max_{t\in[0,1]}\|\mathbf{b}(t)\|-1=\max_{t\in[0,1]}\{\sqrt{\psi(t)+1}-1\}=\sqrt{\|\psi\|_{[0,1]}+1}-1.$$

By the same way, we also get Eq. (2.7) for $\mathbf{b}\in\mathscr{W}_n^{k-}$. □

For $\mathbf{b}\in\mathscr{W}_n^{k}$, we cannot express $\mathscr{H}(\mathbf{q},\mathbf{b})$ in terms of $\|\psi\|_{[0,1]}$ as in Eq. (2.6) or Eq. (2.7), but we can show that $\mathscr{H}(\mathbf{q},\mathbf{b})=\|\sqrt{\psi+1}-1\|_{[0,1]}$.

Definition 2.4. A regular curve $\mathbf{b}$ is said to be *admissible* with respect to $\mathbf{q}$ if and only if

(i) $\mathbf{b}(0)=\mathbf{q}(0)$ and $\mathbf{b}(1)=\mathbf{q}(1)$;

(ii) there exists a unique strictly increasing bijective map $\phi_{\mathbf{b}}:[0,1]\to[0,1]$ such that, for each $s\in[0,1]$, the point $\mathbf{b}(\phi_{\mathbf{b}}(s))$ lies on the normal line $N(s):=\mathbf{q}(s)+u\cdot\mathbf{n}_{\mathbf{q}}(s)$ $(u\in\mathbb{R})$ of $\mathbf{q}$ at $\mathbf{q}(s)$, where $\mathbf{n}_{\mathbf{q}}(s)$ denotes the unit normal vector of the planar curve $\mathbf{q}$ at $\mathbf{q}(s)$;

(iii) the tangent vector of $\mathbf{b}$ at $t=\phi_{\mathbf{b}}(s)$ is not parallel to $\mathbf{n}_{\mathbf{q}}(s)$ for any $s\in[0,1]$.

(Refer to [6], or to [2] for an equivalent definition.)

Let $\mathscr{B}_n^k$ be the class of all admissible Bézier curves with respect to $\mathbf{q}$ of degree n with the contact order $k\geqslant 0$ at each end points of $\mathbf{q}(s)$. We also define $\mathscr{B}_n^{k\pm}$ as in the case of $\mathscr{W}_n^{k\pm}$.

Remark. $\mathscr{B}_n^k\subset\mathscr{W}_n^k$ and $\mathscr{B}_n^{k\pm}\subset\mathscr{W}_n^{k\pm}$.

Proposition 2.5. *If* $\theta_0=0<\theta_1<\cdots<\theta_{n-1}<\theta_n=\cos\alpha$, *where* θ_i *is the angle of the control points* $\mathbf{b}_i$, *then* $\mathbf{b}\in\mathscr{W}_n^0$ *is admissible with respect to the circular arc* $\mathbf{q}(s)$ *as defined in Eq.* (2.1).

Proof. Since $\mathbf{b}\in\mathscr{W}_n^0$, $\mathbf{b}(0)=\mathbf{q}(0)$ and $\mathbf{b}(1)=\mathbf{q}(1)$. By de Casteljau algorithm (see [7]), if $t_{i_0}<t_{i_1}\in[0,1]$, then

$$\arctan\frac{y_{i_0}}{x_{i_0}}<\arctan\frac{y_{i_1}}{x_{i_1}}.$$

Thus, for each $s \in [0,1]$, there exists a unique point $\mathbf{b}(t)$ whose angle equals to that of $\mathbf{q}(s)$. Hence the well-defined map $\phi_{\mathbf{b}}(s)$, for $s \in [0,1]$, is an increasing bijection and satisfies (ii) in Definition 2.4.

By de Casteljau algorithm, for each $t \in [0,1]$, the tangent line of $\mathbf{b}(t)$ exists and cannot pass through the origin. Thus, $\mathbf{b}'(t) \neq \mathbf{0}$, that is, $\mathbf{b}$ is regular, and for each $s \in [0,1]$, the tangent line of $\mathbf{b}(t)$ at $t = \tau(s)$ is not parallel to the line $N(s)$. □

3. Quartic polynomial approximations to the circular arcs

In this section we find the quartic approximation $\mathbf{b}(t)$ to the circular arc of angle $0 < \alpha < \pi$. We put the both end points of the quartic Bézier curve $\mathbf{b}(t)$ as $\mathbf{b}_0 = \mathbf{q}(0)$ and $\mathbf{b}_4 = \mathbf{q}(1)$. To find the remaining control points $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$, we express those as follows.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ u \end{bmatrix}, \quad \mathbf{b}_2 = r \begin{bmatrix} \cos \frac{\alpha}{2} \\ \sin \frac{\alpha}{2} \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix} + u \begin{bmatrix} \sin \alpha \\ -\cos \alpha \end{bmatrix}. \tag{3.1}$$

Then $\mathbf{b}(t)$ and $\mathbf{q}(s)$ have the same tangent direction at both end points and the error function is given by

$$\psi_{\mathbf{b}}(t) = \sum_{i=2}^{6} c_i \binom{8}{i} t^i (1-t)^{8-i},$$

where $c_{8-i} = c_i$. By Lemma 2.2, $\mathbf{b}(t)$ is the GC^2 approximation to the arc if and only if the control coefficient c_2 is zero. By a simple calculation, we have

$$c_2 = \tfrac{1}{14}\{4u^2 + 3r\cos\tfrac{1}{2}\alpha - 3\},$$

so that $\mathbf{b}$ and $\mathbf{q}$ have contact order two if and only if

$$u = \frac{1}{2}\sqrt{3\left(1 - r\cos\frac{\alpha}{2}\right)} \quad \text{or} \quad r = \frac{3 - 4u^2}{3\cos(\alpha/2)}. \tag{3.2}$$

Thus $\mathbf{b}_u \in \mathscr{W}_4^2$ has its signed error function

$$\psi_{\mathbf{b}_u}(t) = \sum_{i=3}^{5} c_i \binom{8}{i} t^i (1-t)^{8-i}, \tag{3.3}$$

where the coefficients $c_3 = c_5$ and c_4 depend on the variable u for the fixed α. In Fig. 1, for $\alpha = \pi/2$, we plot the signed error function $\psi_{\mathbf{b}_u}(t)$, $t \in [0,1]$, of the quartic GC^2 approximation $\mathbf{b}_u(t)$ determined by the variable u. Near $u = 0.4$, the error function $\psi_{\mathbf{b}_u}(t)$ of $\mathbf{b}_u(t)$ is close to the zero function as shown in Table 1 and Fig. 1.

3.1. GC^3 quartic polynomial approximations

In this section we find the GC^3 quartic approximation $\mathbf{b}_u$ to the circular arc. By Lemma 2.2, $\mathbf{b}_u(t)$ having Eq. (3.3) is the GC^3 approximation if and only if the control coefficient c_3 is zero. By a

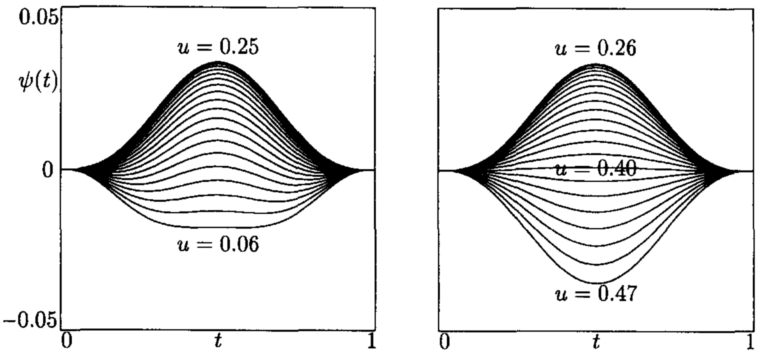

Fig. 1. The signed error function $\psi(t)$, $t \in [0,1]$, of the quartic approximation $\mathbf{b}_u(t)$ determined by u from $u=0.06$ to $u=0.25$ with stepsize 0.01 in the left figure, and from $u=0.26$ to $u=0.47$ in the right figure, for $\alpha=\pi/2$.

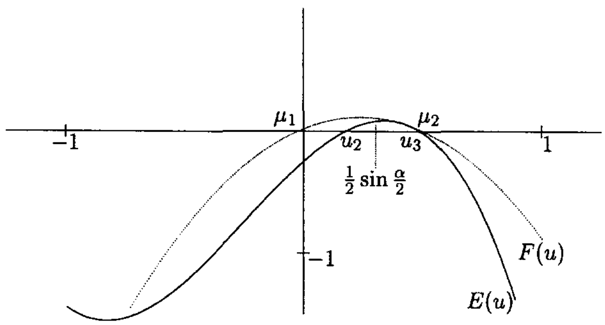

Fig. 2. The cubic polynomial $E(u)$ and the quadratic polynomial $F(u)$ for $\alpha = 0.6\pi$.

simple calculation, we have

$$c_3 = \tfrac{1}{7}\left\{\left(\cos\alpha + u\sin\alpha + 2 - 3r\cos\tfrac{1}{2}\alpha\right) + 6u\left(r\sin\tfrac{1}{2}\alpha - 2u\right)\right\},$$

and by substituting Eq. (3.2) in the above equation, we get

$$c_3 = -\tfrac{8}{7}\tan\tfrac{1}{2}\alpha\left\{u^3 + (\cot\tfrac{1}{2}\alpha)u^2 + (\tfrac{1}{4}\sin^2\tfrac{1}{2}\alpha - 1)u + \frac{\sin\alpha}{8}\right\}. \tag{3.4}$$

For each $0<\alpha<\pi$, we define $E(u)$ to be the cubic polynomial on the right-hand side of Eq. (3.4) and $E_1(u)$ to be its monic polynomial in the braces of Eq. (3.4). That is,

$$c_3 = E(u) := -\tfrac{8}{7}\tan\tfrac{1}{2}\alpha E_1(u). \tag{3.5}$$

Since $0<\alpha<\pi$, E has a zero at u if and only if E_1 has also a zero at u. The quartic Bézier curve $\mathbf{b}_u$ determined by the positive zero u of E_1 is the GC^3 approximation to the circular arc of angle α. Since $E_1(0) > 0$ and $E_1(\frac{1}{2}\sin\frac{1}{2}\alpha) = -\sin\frac{\alpha}{2}\sin^4\frac{\alpha}{4} < 0$ for all $0<\alpha<\pi$, Equation $E_1(u)=0$ has one negative real root u_1 and two positive real roots, u_2 and u_3, satisfying $u_1<0<u_2<\frac{1}{2}\sin\alpha/2<u_3$, as shown in Fig. 2.

Theorem 3.1. *There exist only two GC^3 approximations to the unit circular arc of angle α in the quartic polynomial curves. Both quartic GC^3 approximations are not only in $\mathscr{W}_4^3$ but also admissible to the circular arc.*

Proof. For each positive solution u_j of $E_1(u)$, $j=2,3$, there exists a quartic Bézier curve with the control points $\mathbf{b}_i$, $i=0,\dots,4$, satisfying Eqs. (3.1) and (3.2). By Lemma 2.2, the quartic Bézier curve $\mathbf{b}_{u_j}$ determined by u_j, $j=2,3$, is the GC^3 approximation to the circular arc.

Since

$$u_2 < \frac{\sqrt{3}}{2}\sin\frac{\alpha}{2} \quad \text{and} \quad E_1\left(\frac{\sqrt{3}}{2}\sin\frac{\alpha}{2}\right) = \frac{1}{2}\sin\alpha\left(1-\frac{\sqrt{3}}{2}\cos\frac{\alpha}{2}\right) > 0,$$

we have

$$u_3 < \frac{\sqrt{3}}{2}\sin\frac{\alpha}{2} \quad \text{for all } 0<\alpha<\pi.$$

Thus, by Eq. (3.2), $r>0$ and $\theta_2=\alpha/2$. Since $u_2<u_3<\tan(\alpha/2)$, $\theta_0=0<\theta_1<\theta_2=\alpha/2$. By the symmetry of $\mathbf{b}(t)$ and by Proposition 2.5, both quartic GC^3 approximations are not only in $\mathscr{W}_4^3$ but also admissible . □

Using the Cardan formula [13], we find the explicit form of the roots u_j, $j=1,2,3$, of the cubic polynomial $E(u)=0$ and show that the roots u_j, $j=2,3$, are the distinct positive real roots of $E(u)=0$ in the following proposition.

Proposition 3.2 (Explicit form). *For each $0<\alpha<\pi$, the explicit form of the roots u_j, $j=1,2,3$, of $E(u)=0$ is given by*

$$u_j = 2\cos\left(\phi_\alpha + \frac{2j\pi}{3}\right)\sqrt{S_1(\alpha)} - \frac{1}{3}\cot\frac{\alpha}{2}, \tag{3.6}$$

where

$$\begin{aligned} S_1(\alpha) &:= \frac{1}{36}\csc^2\frac{\alpha}{2}\left(4+8\sin^2\frac{\alpha}{2}-3\sin^4\frac{\alpha}{2}\right),\\ S_2(\alpha) &:= \frac{1}{54}\cos\frac{\alpha}{2}\csc^3\frac{\alpha}{2}\left(4+14\sin^2\frac{\alpha}{2}+9\sin^4\frac{\alpha}{2}\right), \qquad (3.7)\\ \phi_\alpha &:= \frac{1}{3}\arccos\left\{-S_2(\alpha)/2\sqrt{S_1(\alpha)^3}\right\}, \end{aligned}$$

where the range of arccos *function is $[0,\pi]$ and $u_1<0<u_2<u_3$.*

Proof. Note that $S_i(\alpha)>0$, for $i=1,2$ and all $0<\alpha<\pi$, and that ϕ_α is well defined since

$$(2\sqrt{S_1(\alpha)^3})^2 - S_2(\alpha)^2 = \frac{1}{432}\sin^2\frac{\alpha}{2}\left(8+20\sin^2\frac{\alpha}{2}-\sin^4\frac{\alpha}{2}\right) > 0.$$

First, we check that u_j, $j=1,2,3$, are the roots of the cubic polynomial $E_1(u)$. By a simple manipulation of $E_1(u)$, $E_1(u_j)$ is given as follows:

$$\begin{aligned}E_1(u_j) &= \left(u_j + \frac{1}{3}\cot\frac{\alpha}{2}\right)^3 - 3S_1(\alpha)\left(u_j + \frac{1}{3}\cot\frac{\alpha}{2}\right) + S_2(\alpha),\\ &= 2\sqrt{S_1(\alpha)^3}\left\{4\cos^3\left(\phi_\alpha + \frac{2j\pi}{3}\right) - 3\cos\left(\phi_\alpha + \frac{2j\pi}{3}\right)\right\} + S_2(\alpha)\\ &= 2\sqrt{S_1(\alpha)^3}\cos(3\phi_\alpha) + S_2(\alpha) = 0.\end{aligned}$$

Now, we show that u_j, $j=1,2,3$, expressed in Eq. (3.6) are distinct and $u_1<0<u_2<u_3$. Since $\frac{1}{6}\pi<\phi_\alpha<\frac{1}{3}\pi$, we have

$$\cos(\phi_\alpha + \tfrac{2}{3}\pi) < 0 < \cos(\phi_\alpha + \tfrac{4}{3}\pi) < \tfrac{1}{2} < \cos(\phi_\alpha + \tfrac{6}{3}\pi), \tag{3.8}$$

so that u_j, $j=1,2,3$, are distinct real roots of $E_1(u)$. Eqs. (3.6)–(3.8) yield $u_1<u_2<u_3$, and it follows from the observation before Theorem 3.1 that u_2 and u_3 are the positive real roots of $E(u)$. □

For $\mathbf{b}_{u_2}$ and $\mathbf{b}_{u_3}$, the control coefficients c_i, $i \neq 4$, are zero in Eq. (3.3). Thus $\psi(t)=Ct^4(1-t)^4$ for some coefficient C. Since the leading coefficient of ψ is nonnegative, $C>0$ and $\psi(t)>0$ for all $t\in(0,1)$. Hence $\mathbf{b}_{u_j}$, $j=2,3$, lie in $\mathscr{W}_4^{3+}$ and $\|\psi\|_{[0,1]}=\psi(1/2)$.

Theorem 3.3 (Distance). *Each GC^3 approximation $\mathbf{b}_{u_j}$, $j=2,3$, to the arc lies in $\mathscr{W}_4^{3+}$ and its Hausdorff distance from the arc is*

$$\mathscr{H}(\mathbf{q},\mathbf{b}_{u_j}) = \frac{1}{8\cos(\alpha/2)}\left(-4u_j^2 + 2u_j\sin\alpha + 2\sin^2\frac{\alpha}{4}\left(3 - 5\cos\frac{\alpha}{2}\right)\right). \tag{3.9}$$

Proof. Since $\mathbf{b}_{u_j}\in\mathscr{W}_4^{3+}$, $j=2,3$, Proposition 2.3 yields

$$\mathscr{H}(\mathbf{q},\mathbf{b}_{u_j}) = \sqrt{\psi(1/2)+1} - 1 = \sqrt{x(1/2)^2 + y(1/2)^2} - 1. \tag{3.10}$$

By $y(\frac{1}{2}) = x(\frac{1}{2})\tan(\frac{1}{2}\alpha)$ and $x(\frac{1}{2}) = \frac{1}{8}(-4u_j^2 + 2u_j\sin\alpha + 5\cos^2\frac{1}{2}\alpha + 3)$, the assertion follows. □

For each $0<\alpha<\pi$, we define a quadratic polynomial

$$F(u) := \frac{1}{8\cos(\alpha/2)}\left(-4u^2 + 2u\sin\alpha + 2\sin^2\frac{\alpha}{4}\left(3 - 5\cos\frac{\alpha}{2}\right)\right). \tag{3.11}$$

For u_j, $j=2,3$, $F(u_j)$ is equal to the Hausdorff distance between $\mathbf{q}$ and $\mathbf{b}_{u_j}$. In the following theorem, we determine the quartic GC^3 approximation with the smaller error among $\mathbf{b}_{u_j}$, $j=2,3$, using the polynomial F in Eq. (3.11).

Theorem 3.4 (Best approximation). *The Bézier curve $\mathbf{b}_{u_3}$ determined by u_3 is the best quartic GC^3 approximation to the circular arc of angle α.*

Proof. For all $0<\alpha<\pi$, $u_1<0<u_2<\frac{1}{2}\sin\frac{1}{2}\alpha<u_3$. The cubic polynomial $E_2(u)$ having zeros $(u_1+u_2)/2$, $(u_2+u_3)/2$ and $(u_3+u_1)/2$ is shown to be

$$u^3+\cot\frac{\alpha}{2}u^2+\frac{1}{4}\left\{\cot^2\frac{\alpha}{2}-\frac{1}{4}\left(3+\cos^2\frac{\alpha}{2}\right)\right\}u-\frac{1}{8}\left\{\frac{1}{4}\left(3+\cos^2\frac{\alpha}{2}\right)\cot\frac{\alpha}{2}+\frac{\sin\alpha}{8}\right\}.$$

Since $(u_3+u_1)/2$ is the largest root of $E_2(u)=0$ and $E_2(\frac{1}{4}\sin\alpha)<0$, we have $\frac{1}{4}\sin\alpha<(u_2+u_3)/2$ so that $|u_3-\frac{1}{4}\sin\alpha|>|\frac{1}{4}\sin\alpha-u_2|$. Since the quadratic polynomial $F(u)$ has its unique maximum $F(\frac{1}{4}\sin\alpha)$, $F(u_2)>F(u_3)>0$ and the assertion follows. $\square$

In the following theorem, we show that the approximation order of $\mathbf{b}_{u_3}$ is eight.

Theorem 3.5 (Approximation order). *The asymptotic behavior of the Hausdorff distance between $\mathbf{q}$ and $\mathbf{b}_{u_3}$ is*

$$\frac{17-12\sqrt{2}}{2^{15}}\alpha^8+\mathcal{O}(\alpha^{10}).$$

Proof. ϕ_α in Eq. (3.7) has the expansion

$$\phi_\alpha=\frac{\pi}{3}-\frac{\sqrt{6}}{64}\alpha^4+\frac{29}{512\sqrt{6}}\alpha^6+\mathcal{O}(\alpha^8).$$

and so u_3 in Eq. (3.6) has the expansion

$$u_3=\frac{\alpha}{4}+\frac{-4+3\sqrt{2}}{96}\alpha^3+\frac{106-75\sqrt{2}}{7680}\alpha^5+\mathcal{O}(\alpha^7).$$

Therefore, the Hausdorff distance in Eq. (3.9) has the expansion in α as follows.

$$\mathcal{H}(\mathbf{q},\mathbf{b}_{u_3})=\frac{17-12\sqrt{2}}{2^{15}}\alpha^8+\mathcal{O}(\alpha^{10}).\qquad\square$$

3.2. Best quartic $GC^{2\pm}$ approximation

In this section we find the best quartic GC^{2+} or GC^{2-} approximation $\mathbf{b}_u$ whose error function $\psi_{\mathbf{b}_u}(t)\geqslant 0$ or $\psi_{\mathbf{b}_u}(t)\leqslant 0$ for all $t\in[0,1]$, respectively. The GC^2 approximation $\mathbf{b}_u$ satisfies Eqs. (3.1)–(3.3) with $c_3=c_5$. Thus the error function

$$\psi_{\mathbf{b}_u}(t)=14t^3(1-t)^3[4c_3\{(1-t)^2+t^2\}+5c_4t(1-t)] \tag{3.12}$$

has the nonnegative leading coefficient $14(5c_4-8c_3)\geqslant 0$, i.e., $c_4\geqslant\frac{8}{5}c_3$.

Lemma 3.6. (a) $\mathbf{b}\in\mathcal{W}_4^{2+}$ *if and only if c_3 is nonnegative in Eq.* (3.12).

(b) *If* $\mathbf{b} \in \mathscr{W}_4^{2+}$, *then* $\|\psi_{\mathbf{b}}\|_{[0,1]} = \psi_{\mathbf{b}}(\frac{1}{2})$ *and its Hausdorff distance* $\mathscr{H}(\mathbf{q}, \mathbf{b})$ *is equal to* $F(u)$ *for the fixed* α.

(c) $\mathbf{b} \in \mathscr{W}_4^{2-}$ *if and only if* $\psi_{\mathbf{b}}(\frac{1}{2}) \leqslant 0$.

Proof. (a) Assume $c_3 < 0$. Near $t=0$, $\psi(t) = 56c_3t^3 + \mathscr{O}(t^4)$ is negative, i.e., $\mathbf{b} \notin \mathscr{W}_4^{2+}$. Conversely, if $c_3 \geqslant 0$, then $c_4 \geqslant \frac{8}{5}c_3$ and so $\psi(t) \geqslant 0$ by Eq. (3.12).

(b) If $\mathbf{b} \in \mathscr{W}_4^{2+}$, then $c_j \geqslant 0$, $j=3,4$ and by $c_4 \geqslant \frac{8}{5}c_3$, the quadratic polynomial $4c_3\{(1-t)^2+t^2\} + 5c_4t(1-t)$ is nonnegative and has its maximum at $t=\frac{1}{2}$. Thus $\|\psi_{\mathbf{b}}\|_{[0,1]} = \psi_{\mathbf{b}}(\frac{1}{2})$ and it follows from Eqs. (3.9)–(3.11) that $\mathscr{H}(\mathbf{q},\mathbf{b}) = F(u)$.

(c) If $\psi_{\mathbf{b}}(\frac{1}{2}) \leqslant 0$, Eq. (3.12) yields

$$\psi_{\mathbf{b}}\left(\frac{1}{2}\right) = \frac{7}{16}\left(c_3 + \frac{5}{8}c_4\right) \geqslant \frac{7}{8}c_3,$$

and

$$\psi_{\mathbf{b}}(t) = 56t^3(1-t)^3\left[c_3(1-2t)^2 + \frac{32}{7}\psi_{\mathbf{b}}\left(\frac{1}{2}\right)t(1-t)\right] \leqslant 0 \tag{3.13}$$

for all $t \in [0,1]$. Conversely, if $\mathbf{b} \in \mathscr{W}_4^{2-}$, then $\psi_{\mathbf{b}}(\frac{1}{2}) \leqslant 0$. □

Using this lemma, we first find the best GC^{2+} approximation $\mathbf{b}_u$.

Proposition 3.7. *The quartic polynomial curve* $\mathbf{b}_{u_3}$ *obtained in Proposition* 3.2 *is the best approximation from* $\mathscr{W}_4^{2+}$ *and admissible to the circular arc.*

Proof. By Lemma 3.6(a), $\mathbf{b}_u \in \mathscr{W}_4^{2+}$ if and only if

$$c_3 = E(u) = \frac{-8}{7}\tan\frac{\alpha}{2}\,E_1(u) \geqslant 0.$$

As shown in Fig. 2, $\mathbf{b}_u \in \mathscr{W}_4^{2+}$ if and only if $u_2 \leqslant u \leqslant u_3$. Since $F(u)$ has the minimum at $u=u_3$ in the closed interval $[u_2,u_3]$, $\mathbf{b}_{u_3}$ is also the best approximation from $\mathscr{W}_4^{2+}$ to the circular arc. □

Now, we find the best GC^{2-} approximation $\mathbf{b}_u$. By Lemma 3.6, $\mathbf{b}_u \in \mathscr{W}_4^{2-}$ if and only if $F(u) \leqslant 0$. For each α, two real roots, say μ_1 and μ_2, of the quadratic polynomial $F(u)$ is given by

$$\begin{aligned}\mu_1 &= \frac{1}{4}\sin\alpha - \sin^3\frac{\alpha}{4}\sqrt{6+2\cos\frac{\alpha}{2}};\\ \mu_2 &= \frac{1}{4}\sin\alpha + \sin^3\frac{\alpha}{4}\sqrt{6+2\cos\frac{\alpha}{2}}.\end{aligned} \tag{3.14}$$

so that we have a subset $U := U_1 \cup U_2$ of $\mathbb{R}$ defined by

$$U_1 = \{0 < u \leqslant \mu_1 : \mathbf{b}_u \in \mathscr{W}_4^{2}\} \quad \text{and} \quad U_2 = \{u \geqslant \mu_2 : \mathbf{b}_u \in \mathscr{W}_4^{2}\}.$$

Note that, for $0 < 2\arccos\frac{3}{5} < \alpha < \pi$, $\mu_1 < 0$ and U_1 is empty.

Proposition 3.8. *The quartic GC^2 approximations $\mathbf{b}_{\mu_j}$, $j=1,2$, are not only in $\mathscr{W}_4^{2-}$ but also admissible to the circular arc.*

Proof. Eq. (3.14) yields $\mu_j < \tan\alpha/2$ and $\mu_j < \sqrt{3}/2$ for all $0<\alpha<\pi$ and $j=1,2$. Thus, by Eq. (3.2), $r>0$, and by Proposition 2.5, both approximations $\mathbf{b}_{\mu_j}$, $j=1,2$, lie not only in $\mathscr{W}_4^{2-}$ but also in $\mathscr{B}_4^{2-}$. □

Lemma 3.9. *Each quartic Bézier curve $\mathbf{b}_{\mu_j}$, $j=1,2$, is the best approximation from $\{\mathbf{b}_u : u \in U_j\}$ to the circular arc.*

Proof. Since $c_3 = E(u)$ and $\psi(\frac{1}{2}) = (F(u)+1)^2 - 1$ are increasing with respect to u in U_1 and decreasing in U_2, so is $\psi_{\mathbf{b}_u}(t)$ by Eq. (3.13), i.e., $\psi_{\mathbf{b}_u}(t) \leqslant \psi_{\mathbf{b}_{\mu_j}}(t) \leqslant 0$, for all $t\in[0,1]$. Thus $\mathbf{b}_{\mu_j}$ is the best approximation from $\{\mathbf{b}_u : u\in U_j\}$, for $j=1,2$. □

Theorem 3.10 (Best approximation and its Hausdorff distance). *For each $0<\alpha<\pi$, $\mathbf{b}_{\mu_2}$ is the best approximation from $\mathscr{W}_4^{2-}$ to the circular arc of angle α, and its Hausdorff distance from the circular arc $\mathbf{q}$ is*

$$\mathscr{H}(\mathbf{q}, \mathbf{b}_{\mu_2}) = 1 - \sqrt{1 + \frac{7\times 27}{2^{11}} E(\mu_2)}, \tag{3.15}$$

where $E(u)$ is the cubic polynomial in Eq. (3.4).

Proof. For $\mathbf{b}_{\mu_j}$, $j=1,2$, $\psi(\frac{1}{2})=0$ and by Eq. (3.13)

$$\psi(t) = 56t^3(1-t)^3(1-2t)^2E(u).$$

Since $\psi'(t) = 56\tan(\frac{1}{2}\alpha)t^2(1-t)^2(1-2t)(1-4t)(3-4t)E(u)$, we get

$$\|\psi\|_{[0,1]} = -\psi\left(\frac{1}{4}\right) = -\frac{7\times 27}{2^{11}}E(u).$$

Since, by Eq. (3.5), we have

$$E(\mu_2) - E(\mu_1) = \frac{8}{7}(\mu_2-\mu_1)\tan\frac{\alpha}{2}\sin^4\frac{\alpha}{4}\left(4\cos^4\frac{\alpha}{4} - 1\right) > 0$$

for each $0<\alpha<\pi$, $\mathbf{b}_{\mu_2}$ is the best approximation from $\mathscr{W}_4^{2-}$ to the circular arc of angle α. □

Note that $E(\mu_2)$ is given by

$$-\frac{32}{7}\sin^8\frac{\alpha}{4}\left(8 + 8\cos\frac{\alpha}{2} + \cos\alpha - \sqrt{6+2\cos\frac{\alpha}{2}}\left(5\cos\frac{\alpha}{4} + \cos\frac{3\alpha}{4}\right)\right). \tag{3.16}$$

Using this equation, we can show that the approximation order of $\mathbf{b}_{\mu_2}$ is also eight.

Theorem 3.11 (Approximation order). *The error of the best quartic GC^2 approximation $\mathbf{b}_{\mu_2}$ from $\mathscr{W}_4^{2-}$ to the circular arc of angle α is*

$$\frac{27(17-12\sqrt{2})}{2^{23}}\alpha^8+\mathcal{O}(\alpha^{10}).$$

Proof. Eq. (3.15) yields

$$\mathscr{H}(\mathbf{q},\mathbf{b}_{\mu_2})=-7\times 27\times 2^{-12}E(\mu_2)+\mathcal{O}(E(\mu_2)^2)$$

and Eq. (3.16) has the expansion

$$E(\mu_2)=(-17+12\sqrt{2})\times 2^{-11}\times 7^{-1}\alpha^8+\mathcal{O}(\alpha^{10}),$$

so that the assertion follows. □

4. Quintic polynomial approximations to the circular arcs

In this section we find the quintic approximation $\mathbf{b}(t)$ to the circular arc of angle α. We put the both end points of the quintic Bézier $\mathbf{b}(t)$ as $\mathbf{b}_0=\mathbf{q}(0)$ and $\mathbf{b}_5=\mathbf{q}(1)$. To find the remaining control points $\mathbf{b}_1,\ldots,\mathbf{b}_4$, we express those as follows.

$$\mathbf{b}_1=\begin{bmatrix}1\\ v\end{bmatrix},\quad \mathbf{b}_2=\begin{bmatrix}\xi\\ \eta\end{bmatrix},\quad \mathbf{b}_3=\begin{bmatrix}\xi\cos\alpha+\eta\sin\alpha\\ \xi\sin\alpha-\eta\cos\alpha\end{bmatrix},\quad \mathbf{b}_4=\begin{bmatrix}\cos\alpha+v\sin\alpha\\ \sin\alpha-v\cos\alpha\end{bmatrix}.$$

Then $\mathbf{b}(t)$ and $\mathbf{q}(s)$ have the same tangent direction at both end points and the error function is given by

$$\psi_{\mathbf{b}}(t)=\sum_{i=2}^{8}c_i\binom{10}{i}t^i(1-t)^{10-i},$$

where $c_{10-i}=c_i$. By Lemma 2.2, $\mathbf{b}(t)$ is the GC^3 approximation to the arc if and only if the control coefficients c_2 and c_3 are zero. By a simple calculation, we have

$$c_2=\frac{1}{18}\{4\xi-4\ +5v^2\};$$

$$c_3=\frac{1}{12}\{\xi\cos\alpha+\eta\sin\alpha-3\xi+2+5v\eta-10v^2\}.$$

so that $\mathbf{b}$ and $\mathbf{q}$ have contact order three if and only if

$$\xi=1-\tfrac{5}{4}v^2, \tag{4.1}$$

$$\eta=\frac{\frac{5}{4}(5+\cos\alpha)v^2+2\sin^2\alpha/2}{J_1(v)}, \tag{4.2}$$

where $J_1(v)=5v+\sin\alpha$. Thus $\mathbf{b}_v(t)\in\mathscr{W}_5^3$ has its signed error function

$$\psi_{\mathbf{b}_v}(t)=\sum_{i=4}^{6}c_i\binom{10}{i}t^i(1-t)^{10-i}, \tag{4.3}$$

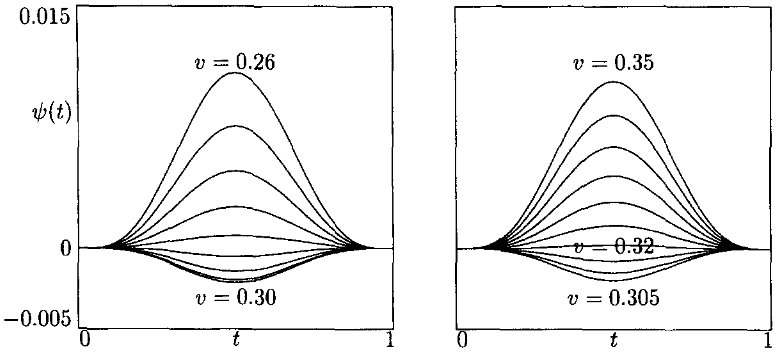

Fig. 3. The signed error function $\psi(t)$, $t \in [0,1]$, of the quintic approximation $\mathbf{b}_v(t)$ determined by v from $v=0.26$ to $v=0.30$ with stepsize 0.005 in the left figure, and from $v=0.305$ to $v=0.35$ in the right figure, for $\alpha=\pi/2$.

where the coefficients $c_4=c_6$ and c_5 depend on the variable v for the fixed α. In Fig. 3, for $\alpha=\pi/2$, we plot the signed error function $\psi_{\mathbf{b}_v}(t)$, $t\in[0,1]$, of the quintic GC^3 Bézier approximation $\mathbf{b}_v(t)$ determined by the variable v. Near $v=0.32$, the error function $\psi_{\mathbf{b}}(t)$ of $\mathbf{b}_v(t)$ is close to the zero function as shown in Table 1 or Fig. 3.

4.1. GC^4 quintic polynomial approximations

We find the GC^4 quintic approximation $\mathbf{b}_v$ to the circular arc in this section. By Lemma 2.2, $\mathbf{b}_v(t)$ having Eq. (4.3) is the GC^4-approximation if and only if the control coefficients c_4 is zero. By a routine calculations, we have

$$\begin{aligned} c_4 = {} & \frac{1}{42}\{10(\xi-1)^2+\cos\alpha+v\sin\alpha+6\xi+1-4(\xi\cos\alpha+\eta\sin\alpha+1) \\ & +10(\eta-2v)^2+10(\xi\sin\alpha-\eta\cos\alpha+3v-3\eta)v\}. \end{aligned}$$

For each $0<\alpha<\pi$, we define a rational function $G(v)$ by

$$c_4=G(v):=\frac{G_1(v)}{336J_1(v)^2}, \tag{4.4}$$

where the polynomial $G_1(v)$ of degree six is given by

$$\begin{aligned} G_1(v) := {} & 32(9-\cos\alpha)\sin^4\frac{\alpha}{2}-16v\sin\alpha\sin^2\frac{\alpha}{2}(49-\cos\alpha) \\ & +40v^2\sin^2\frac{\alpha}{2}(49\cos\alpha-1)+100v^3(26\sin\alpha-5\sin 2\alpha) \\ & +250v^4(-10+\cos 2\alpha-15\cos\alpha)-1250v^5\sin\alpha+3125v^6. \end{aligned} \tag{4.5}$$

Since $0<\alpha<\pi$, G has a zero at the positive v if and only if G_1 has also a zero at v. The quintic Bézier curve $\mathbf{b}_v$ determined by the zero v of G_1 is the GC^4-approximation to the circular arc of angle α. Using the series expansion of $G_1(v)$ in α we have two roots of $G_1(v)$ near $v=\frac{1}{5}\alpha$ by a trial and error.

Theorem 4.1 (Existence). *For each $0<\alpha<\pi$, there exist at least two quintic GC^4 approximations to the circular arc of angle α. The quintic GC^4 approximations lie not only in $\mathscr{W}_5^4$ but also admissible to the circular arc.*

Proof. Since

$$G_1\left(\frac{2}{5}\sin\frac{\alpha}{2}\right)>0, \quad G_1\left(\frac{4}{5}\sin\frac{\alpha}{4}\right)<0 \quad \text{and} \quad G_1\left(\frac{4}{5}\tan\frac{\alpha}{4}\right)<0 \quad \text{for all } 0<\alpha<\pi,$$

there exist at least two real roots of $G_1(v)=0$, say

$$\frac{2}{5}\sin\frac{\alpha}{2}<v_1<\frac{4}{5}\sin\frac{\alpha}{4}<v_2<\frac{4}{5}\tan\frac{\alpha}{4}. \tag{4.6}$$

For $0<v<\frac{4}{5}\tan\frac{1}{4}\alpha$, Eq. (4.1) yields

$$0<1-\frac{4}{5}\tan^2\frac{\alpha}{4}<\xi<1,$$

and so Eq. (4.2) yields

$$\eta-v>\frac{4}{5}\sin\frac{\alpha}{2}\tan\frac{\alpha}{4}\left(5\cos^2\frac{\alpha}{4}-\cos\frac{\alpha}{2}\right)J(v)^{-1}>0, \tag{4.7}$$

$$\xi\tan\frac{\alpha}{2}-\eta=-\frac{25}{4}\sin\frac{\alpha}{2}J_1(v)^{-1}v\left(v^2+2\cot\frac{\alpha}{2}v-\frac{4}{5}\right)>0.$$

Thus for $j=1,2$, $0<v_j<\eta_j/\xi_j<\tan\frac{1}{2}\alpha$. Hence each quintic GC^4-approximation $\mathbf{b}_{v_j}$, $j=1,2$, determined by v_j, lies not only in $\mathscr{W}_5^4$ but also in $\mathscr{B}_5^4$, by Proposition 2.5. □

For $\mathbf{b}_{v_j}$, $j=1,2$, the control coefficients c_i, $i\neq 5$, are zero in Eq. (4.3). Thus $\psi(t)=Ct^5(1-t)^5$ for some constant C. Since the leading coefficient of ψ is nonnegative, $C<0$ and $\psi(t)<0$ for all $t\in(0,1)$. Hence $\mathbf{b}_{v_j}$, $j=1,2$, lies in $\mathscr{W}_5^{4-}$ and $\|\psi\|_{[0,1]}=|\psi(1/2)|$.

Theorem 4.2 (Distance). *The quintic GC^4 approximation $\mathbf{b}_{v_j}$, $j=1,2$, to the arc lies in $\mathscr{W}_5^{4-}$ and its Hausdorff distance is*

$$\mathscr{H}(\mathbf{q},\mathbf{b}_{v_j})=-H_1(v_j)/32J_1(v_j), \tag{4.8}$$

where $J_1(v)=5v+\sin\alpha$ and the cubic polynomial $H_1(v)$ is given by

$$H_1(v):=\left(-125v^3\cos\frac{\alpha}{2}+80\sin^2\frac{\alpha}{4}\left(-3-3\sin^2\frac{\alpha}{4}+2\sin^4\frac{\alpha}{4}\right)v\right.$$
$$\left.+150v^2\sin\frac{\alpha}{2}+32\sin\frac{\alpha}{2}\sin^2\frac{\alpha}{4}\left(4-3\cos^2\frac{\alpha}{4}\right)\right). \tag{4.9}$$

Proof. Since $\mathbf{b}_{v_j}\in\mathscr{W}_5^{4-}$, $j=1,2$, by Proposition 2.3

$$\mathscr{H}(\mathbf{q},\mathbf{b}_{v_j})=1-\sqrt{\psi(1/2)+1}=1-\sqrt{x(1/2)^2+y(1/2)^2}.$$

By the same way in the proof of Theorem 3.3, we have Eq. (4.8). □

Theorem 4.3 (Approximation order). *The error of GC^4 approximation to the circular arc of angle α by the quintic polynomial curves $\mathbf{b}_{v_j}$, $j=1,2$, is $\mathcal{O}(\alpha^{10})$.*

Proof. By the series expansion, Eq. (4.6) yields

$$v_j = \frac{\alpha}{5} + v_{j,3}\,\alpha^3 + \mathcal{O}(\alpha^5) \quad \text{for } j=1,2,$$

and Eq. (4.5) yields $G_1(v_j)=g(v_{j,3})\,\alpha^{10} + \mathcal{O}(\alpha^{12})$, where the cubic polynomial $g(v)$ is given by $g(v)= -3200v^3 + 160v^2 + \frac{4}{3}v - \frac{1}{540}$. Since v_j, $j=1,2$, is a root of $G_1(v)$, $v_{j,3}$, $j=1,2$, is also a root of $g(v)$. The assertion follows from the expansion $\mathscr{H}(\mathbf{q},\mathbf{b}_{v_j})=5\times 2^{-13}\,g(v_{j,3})\,\alpha^8 + \mathcal{O}(\alpha^{10})$ of Eq. (4.8). □

4.2. Best $GC^{3\pm}$ quintic approximation

In this section we find the best quintic GC^{3+} or GC^{3-} approximation $\mathbf{b}_v$ whose error function $\psi_{\mathbf{b}_v}(t)\geqslant 0$ or $\psi_{\mathbf{b}_v}(t)\leqslant 0$ for all $t\in[0,1]$, respectively. The GC^3 approximation $\mathbf{b}_v$ satisfies Eqs. (4.1) and (4.2) with $c_4=c_6$. Thus the error function

$$\psi(t) = 42\,t^4(1-t)^4[5c_4\{(1-t)^2+t^2\} + 6c_5t(1-t)] \tag{4.10}$$

has the nonnegative leading coefficient $42(10c_4-6c_5)$, i.e., $c_4\geqslant\frac{3}{5}c_5$.

Lemma 4.4. (a) $\mathbf{b}\in\mathscr{W}_5^{3+}$ *if and only if* $\psi_{\mathbf{b}}(\frac{1}{2})\geqslant 0$.
(b) $\mathbf{b}\in\mathscr{W}_5^{3-}$ *if and only if c_4 is nonpositive in Eq.* (4.10).

Proof. Using the method in the proof of Lemma 3.6 and

$$\psi_{\mathbf{b}}(t)=42t^4(1-t)^4\left[5c_4(1-2t)^2 + \frac{512}{21}\psi_{\mathbf{b}}\left(\frac{1}{2}\right)t(1-t)\right] \tag{4.11}$$

we obtain the assertions. □

Using this lemma, we first find the best GC^{3+} approximation $\mathbf{b}_v$. For each $0<\alpha<\pi$, $\mathbf{b}_v\in\mathscr{W}_5^{3+}$ if and only if $H_1(v)\geqslant 0$, by the above lemma and Eq. (4.8). Since the cubic polynomial $H_1(v)$ is factorized by $(5v-2\sin\frac{1}{2}\alpha)H_2(v)$, where the quadratic polynomial $H_2(v)$ is given by

$$-25v^2\cos\frac{\alpha}{2} + \left(30\sin\frac{\alpha}{2} - 5\sin\alpha\right)v - 16\sin^2\frac{\alpha}{4}\left(4-3\cos^2\frac{\alpha}{4}\right), \tag{4.12}$$

we can find all roots of $H_1(v)$ as in the following:

$$\begin{aligned} v_1 &= \frac{2}{5}\sin\frac{\alpha}{2}, \\ v_2 &= \frac{2}{5}\sin\frac{\alpha}{4}\left(\cos\frac{\alpha}{4}\left(3-\cos\frac{\alpha}{2}\right) - \sqrt{2\left(9+\cos\frac{\alpha}{2}\right)\sin^2\frac{\alpha}{4}}\right)\sec\frac{\alpha}{2}, \end{aligned} \tag{4.13}$$

$$v_3 = \frac{2}{5}\sin\frac{\alpha}{4}\left(\cos\frac{\alpha}{4}\left(3-\cos\frac{\alpha}{2}\right)+\sqrt{2\left(9+\cos\frac{\alpha}{2}\right)\sin^2\frac{\alpha}{4}}\right)\sec\frac{\alpha}{2}.$$

Then

$$\frac{2}{5}\sin\frac{\alpha}{2}<\frac{v_2+v_3}{2} \quad\text{and}\quad H_2\left(\frac{2}{5}\sin\frac{\alpha}{2}\right) = -64\sin^6\frac{\alpha}{4}<0$$

so $v_1<v_2<v_3$. From Eq. (4.12), we have

$$H_2\left(\frac{4}{5}\tan\frac{\alpha}{4}\right) = 16\tan^2\frac{\alpha}{4}\sin^4\frac{\alpha}{4}>0 \quad\text{and}\quad v_2<\frac{4}{5}\tan\frac{\alpha}{4}<v_3.$$

Since $J_1(v_2)>0$, $H_1(v_2)<0$ by Eq. (4.8) so that $\upsilon_2<v_2$. Hence, it follows from Eq. (4.6) that

$$\frac{2}{5}\sin\frac{\alpha}{2}=v_1<\upsilon_1<\frac{4}{5}\sin\frac{\alpha}{4}<\upsilon_2<v_2<\frac{4}{5}\tan\frac{\alpha}{4}<v_3. \tag{4.14}$$

Proposition 4.5. *The quintic Bézier approximations* $\mathbf{b}_{v_j}$, $j=1,2$, *are not only in* $\mathscr{W}_5^{3+}$ *but also admissible to the circular arc.*

Proof. Since $0<v_j<\frac{4}{5}\tan\frac{1}{4}\alpha$ for each $j=1,2$, $\mathbf{b}_{v_j}$ satisfies Eq. (4.7) so that $\mathbf{b}_{v_j}$ lies in $\mathscr{B}_5^3$. Thus the assertion follows from $H_1(v_j)=0$. □

We define a subset $V:=V_1\cup V_2$ of $\mathbb{R}$ by $V_1:=\{0<\upsilon\leqslant v_1:\mathbf{b}_v\in\mathscr{B}_5^{3+}\}$ and $V_2:=\{v_2\leqslant\upsilon\leqslant v_3:\mathbf{b}_v\in\mathscr{B}_5^{3+}\}$. Thus $\mathscr{B}_5^{3+}=\{\mathbf{b}_\upsilon:\upsilon\in V_1\cup V_2\}$.

Lemma 4.6. *The quintic Bézier curve* $\mathbf{b}$ *is a locally best approximation from* $\mathscr{B}_5^{3+}$ *to the arc if and only if* $\psi_{\mathbf{b}}(t)=0$ *for some* $t\in(0,1)$.

Proof. By an application of Theorem 2 in [5], $\mathbf{b}$ is a locally best approximation if and only if $\psi_{\mathbf{b}}(t)=0$ for some $t\in(0,1)$ or $\mathbf{b}\in\mathscr{B}_5^{4+}$. Since $\mathscr{B}_5^{4+}$ is empty, we get the assertion. □

Theorem 4.7 (Best approximation). *Each quintic Bézier curve* $\mathbf{b}_{v_j}$, $j=1,2$, *is the best approximation to the arc from* $\{\mathbf{b}_\upsilon:\upsilon\in V_j\}$. *Thus the quintic curve* $\mathbf{b}_{v_j}$, $j=1$ *or* 2, *is the best approximation to the circular arc from* $\mathscr{B}_5^{3+}$.

Proof. By Lemma 4.6, $\mathbf{b}_\upsilon$ is a locally best approximation from $\{\mathbf{b}_\upsilon:\upsilon\in V_1\}$ or $\{\mathbf{b}_\upsilon:\upsilon\in V_2\}$ if and only if $\psi_{\mathbf{b}_\upsilon}(\frac{1}{2})=0$, that is $\upsilon=v_j$, $j=1,2$ or 3. Since $G(\upsilon)$ is decreasing near $\upsilon=0$ by Eq. (4.4), $\mathbf{b}_{v_1}$ is the best approximation from $\{\mathbf{b}_\upsilon:\upsilon\in V_1\}$. Since, for each $0<\alpha<\pi$

$$\begin{aligned}G(v_3)-G(v_2) &= \frac{64\sqrt{2(9+\cos\frac{\alpha}{2})}}{105J(v_2)^2J(v_3)^2}\left(41\cos\frac{\alpha}{2}+13\cos\frac{3}{4}\alpha+\cos\frac{5}{4}\alpha\right)\\ &\quad\times\cos\frac{\alpha}{4}(51+12\cos\alpha+\cos 2\alpha)^2\sec^6\frac{\alpha}{2}\sin^{14}\frac{\alpha}{4}>0,\end{aligned}$$

$\mathbf{b}_{v_2}$ is the best approximation from $\{\mathbf{b}_v : v \in V_2\}$ whether $\mathbf{b}_{v_3} \in V_2$ or not. Therefore, $\mathbf{b}_{v_j}$, $j=1$ or 2, is the best approximation from $\mathscr{B}_5^{3+}$. $\square$

We could not prove but we guess that $\mathbf{b}_{v_2}$ is the better approximation than $\mathbf{b}_{v_1}$, using the following proposition and MATHEMATICA.

Theorem 4.8 (Distance). *The Hausdorff distance between the circular arc of angle α and the quintic approximation $\mathbf{b}_{v_j}$, $j=1,2$, is*

$$\mathscr{H}(\mathbf{q}, \mathbf{b}_{v_j}) = \sqrt{\frac{42}{635} G(v_j) + 1} - 1, \tag{4.15}$$

where $G(v)$ is the polynomial of degree six in Eqs. (4.4) and (4.5).

Proof. Let $v = v_1$ or v_2. Since $\psi_{\mathbf{b}_v}(1/2) = 0$, $j=1,2$, Eq. (4.11) yields

$$\psi_{\mathbf{b}_v}(t) = 210t^4(1-t)^4(1-2t)^2 G(v),$$

Using $\psi'_{\mathbf{b}_v}(t) = 840t^3(t-1)^3(t-1/2)(5t^2-5t+1)G(v)$, we get

$$\|\psi_{\mathbf{b}_v}\|_{[0,1]} = \psi_{\mathbf{b}_v}\left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right) = \frac{210}{5^5} G(v).$$

By Proposition 2.3 and Proposition 4.5,

$$\mathscr{H}(\mathbf{q}, \mathbf{b}_{v_j}) = \sqrt{\frac{42}{625} G(v_j) + 1} - 1 \quad \text{for } j=1,2. \qquad \square$$

Theorem 4.9 (Approximation order). *The error of GC^2 approximation $\mathbf{b}_{v_j}$, $j=1,2$, to the circular arc of angle α is $C\alpha^{10} + \mathcal{O}(\alpha^{12})$.*

Proof. From Eq. (4.13), we get

$$\begin{aligned} v_1 &= \frac{\alpha}{5} - \frac{\alpha^3}{120} + \frac{\alpha^5}{9600} + \mathcal{O}(\alpha^7), \\ v_2 &= \frac{\alpha}{5} + \frac{(7-3\sqrt{5})\alpha^3}{240} + \frac{(47-21\sqrt{5})\alpha^5}{19200} + \mathcal{O}(\alpha^7). \end{aligned}$$

It follows from Eq. (4.15) and the above equation that

$$\begin{aligned} \mathscr{H}(\mathbf{q}, \mathbf{b}_{v_1}) &= \frac{\alpha^{10}}{256 \times 10^5} + \mathcal{O}(\alpha^{12}); \\ \mathscr{H}(\mathbf{q}, \mathbf{b}_{v_2}) &= \frac{9(123-55\sqrt{5})\alpha^{10}}{512 \times 10^5} + \mathcal{O}(\alpha^{12}). \qquad \square \end{aligned}$$

Now, we find the locally best approximation $\mathbf{b}(t)$ from $\mathscr{W}_5^{3-}$ to the circular arc. By the Lemma 4.4(b), we know that $\mathbf{b}_v$ lies in $\mathscr{W}_5^{3-}$ if and only if $G(v) \leqslant 0$. But we cannot find the intervals of v-variable for which $\mathbf{b}_v$ is contained in $\mathscr{W}_5^{3+}$, because the sixth-degree polynomial inequality equation $G(v) \leqslant 0$ is not solvable algebraically. We can only show the existence of the locally best GC^{3-} approximation, $\mathbf{b}_{v_1}$ or $\mathbf{b}_{v_2}$ by Lemma 4.6.

Proposition 4.10. *The quintic Bézier curve* $\mathbf{b}_{v_j}$, $j = 1, 2$, *obtained in Theorem* 4.1 *is the locally best approximation from* $\mathscr{W}_5^{3-}$ *and admissible to the circular arc.*

5. Conclusion

We presented the explicit forms of the best GC^3, GC^{2+} and GC^{2-} approximation to the circular arc of angle $0 < \alpha < \pi$ from the quartic Bézier curves and found the explicit form of the Hausdorff distance between the circular arc and the Bézier curves for each case. We also presented the existence and the numerical solution of the quintic GC^4 approximations to the arc and found the explicit form of the best GC^{3+} approximation to the arc from the quintic Bézier curves and gave the explicit form of the Hausdorff distance between two curves. We also showed that our approximations by the quartic or quintic Bézier curves have the optimal approximation order eight or ten, respectively. This supports the validity of the conjecture in [11, 15] that nth degree Bézier curve approximation have the optimal approximation order $2n$.

Acknowledgements

The authors are very grateful to the anonymous referees for the inspiring comments and the valuable suggestions which improved our paper considerably.

References

[1] C. de Boor, K. Höllig, M. Sabin, High accuracy geometric Hermite interpolation, Comput. Aided Geom. Des. 4 (1987) 169–178.
[2] W.L.F. Degen, Best approximations of parametric curves by spline, in: T. Lyche, L.L. Schumaker (Eds.), Mathematical Methods in CAGD and Image Processing, Academic Press, New York, 1992, pp. 171–184.
[3] W.L.F. Degen, High accurate rational approximation of parametric curves, Comput. Aided Geom. Design 10 (1993) 293–313.
[4] T. Dokken, M. Dæhlen, T. Lyche, K. Mørken, Good approximation of circles by curvature-continuous Bézier curves, Comput. Aided Geom. Design 7 (1990) 33–41.
[5] E.F. Eisele, Best constrained approximations of planar curves by Bézier curves, in: P.J. Laurent, A. Le Méhauté, L.L. Schumaker (Eds.), Curves and Surfaces in Geometric Design 1994, pp. 139–146.
[6] E.F. Eisele, Chebychev approximation of plane curves by splines, J. Approx. Theory 76 (1994) 133–148.
[7] G. Farin, Curves and Surfaces for Computer Aided Geometric Design, Academic Press, San Diego, CA, 1993.
[8] M. Floater, An $\mathcal{O}(h^{2n})$ Hermite approximation for conic sectons, Comput. Aided Geom. Design 14 (1997) 135–151.
[9] M. Goldapp, Approximation of circular arcs by cubic polynomials, Comput. Aided Geom. Design 8 (1991) 227–238.
[10] J.A. Gregory, Geometric continuity, in: T. Lyche, L.L. Schumaker (Eds.) Mathematical Methods in CAGD, Academic Press, Nashville, 1989, pp. 353–371.
[11] K. Höllig, J. Koch, Geometric Hermite interpolation, Comput. Aided Geom. Design 12 (1995) 567–580.
[12] K. Höllig, J. Koch, Geometric Hermite interpolation with maximal order and smoothness, Comput. Aided Geom. Design 13 (1996) 681–696.

[13] N. Jacobson, Basic Algebra I, W.H. Freeman, New York, 1985.
[14] K. Mørken, Best approximation of circle segments by quadratic Bézier curves, in: P.J. Laurent, A. la Méhauté, L.L. Schumaker (Eds.), Curves and Surfaces, Academic Press, New York, 1990.
[15] A. Rababah, Taylor theorem for planar curves, Proc. Amer. Math. Soc. 119 (1993) 803–810.
[16] R. Schaback, Planar curve interpolation by piecewise conics of arbitrary type, Constr. Approx. 9 (1993) 373–389.